# Stochastic Spectral Descent Methods

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## 1. Introduction

Consider the following optimization problem

$$\min_{x\in\mathbb{R}^n} f(x) := \tfrac{1}{2}x^\top \mathbf{A}x - b^\top x,$$

where $\mathbf{A}$ is an $n\times n$ symmetric positive definite matrix. The problem has a unique solution: $x_* = \mathbf{A}^{-1}b$. We are interested in the case when $n$ is huge (millions, billions). Note that $f$ is (strongly) convex and quadratic.

## 2. Algorithm: Stochastic Descent

The state-of-the-art methods for convex optimization in huge dimensions are randomized coordinate descent (RCD) methods. We now describe a method which includes RCD as a special case: **stochastic descent (SD).** SD is a special case of the **sketch-and-project** method developed in [1].

**Algorithm 1 [1, 3] (Stochastic Descent).**

***Parameter:*** *some distribution $\mathcal{D}$ over vectors in $\mathbb{R}^n$*
***Initialization:*** *Choose $x_0\in\mathbb{R}^n$*
***for*** $t = 0, 1, 2\dots$ ***do***
*Draw a fresh sample $s_t$ from $\mathcal{D}$*
$x_{t+1} \leftarrow x_t - \frac{s_t^\top(\mathbf{A}x_t - b)}{s_t^\top \mathbf{A} s_t} s_t$
***end for***

RCD is obtained as a special case by letting $\mathcal{D}$ be a distribution over unit coordinate (i.e., basis) vectors in $\mathbb{R}^n$: $\{e_1, e_2, \cdots, e_n\}$:

$$s_t \sim \mathcal{D} \quad\Leftrightarrow\quad s_t = e_i \quad \text{with probability} \quad p_i > 0.$$

**Theorem 1 [1, 3].** *Algorithm 1 converges linearly in expectation as*

$$(1-\rho_{\max})^t\|x_0 - x_*\|^2_{\mathbf{A}} \le \mathbb{E}_{s\sim\mathcal{D}}[\|x_t - x_*\|^2_{\mathbf{A}}] \le (1-\rho_{\min})^t\|x_0 - x_*\|^2_{\mathbf{A}},$$

*where* $\|x\|_{\mathbf{A}} = (x^\top\mathbf{A}x)^{1/2}$, $\mathbf{W} := \mathbb{E}_{s\sim\mathcal{D}}\left[\frac{\mathbf{A}^{1/2}ss^\top\mathbf{A}^{1/2}}{s^\top\mathbf{A}s}\right]$, $\rho_{\max} = \lambda_{\max}(\mathbf{W})$, $\rho_{\min} = \lambda_{\min}(\mathbf{W})$. *Moreover,* $0 < \rho_{\min} \le 1/n$ *and* $\rho_{\max}\le 1$.

## 3. Research Question

RCD with probabilities $p_i = \mathbf{A}_{ii}/\mathrm{Tr}(\mathbf{A})$ satisfies: $\rho_{\min} = \lambda_1/\mathrm{Tr}(\mathbf{A})$, where $\lambda_1$ is the smallest eigenvalue of $\mathbf{A}$. When $\rho_{\min}$ is small, RCD is slow. **Can we modify RCD by utilizing some spectral information, if known, so that the rate gets improved?**

## 4. New Algorithm

Let $\mathbf{A} = \sum_{i=1}^n \lambda_i u_i u_i^\top$ be the eigenvalue decomposition of $\mathbf{A}$, with $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_n$ being the eigenvalues, and $u_1,\dots,u_n$ the eigenvectors.

**Algorithm 2 [2] (Stochastic Spectral Coordinate Descent).**

***Parameter:*** *Choose $k\in\{0,\dots,n-1\}$; set $C_k = k\lambda_{k+1} + \sum_{i=k+1}^n \lambda_i$*
*Run Algorithm 1 with the following distribution $\mathcal{D}$:*

$$s_t = \begin{cases} e_i & \text{with probability } p_i = \frac{\mathbf{A}_{ii}}{C_k}, \quad i = 1,2,\dots,n \\ u_i & \text{with probability } p_{n+i} = \frac{\lambda_{k+1}-\lambda_i}{C_k}, \quad i=1,2,\dots,k.\end{cases}$$

Note that for $k=0$, Algorithm 2 reduces to RCD.

**Theorem 2.** *For every $n\ge 2$, Algorithm 2 has the rate*

$$\rho_{\min} = \frac{\lambda_{k+1}}{C_k}.$$

*Moreover, the rate improves as $k$ grows, and interpolates between the RCD rate $\lambda_1/\mathrm{Tr}(\mathbf{A})$ for $k=0$, and the optimal rate $1/n$ for $k=n-1$:*

$$\frac{\lambda_1}{\mathrm{Tr}(\mathbf{A})} = \frac{\lambda_1}{C_0} \le \cdots \le \frac{\lambda_{k+1}}{C_k} \le \cdots \le \frac{\lambda_{n-1}}{C_{n-2}} \le \frac{\lambda_n}{C_{n-1}} = \frac{1}{n}.$$

The total work of Algorithm 2 depends on $k$:

$$Work(\mathcal{D}) := \underbrace{P(\mathcal{D})}_{\text{preprocessing cost}} + \underbrace{C(\mathcal{D})}_{\text{cost of 1 iteration}} \times \underbrace{I(\mathcal{D})}_{\text{number of iterations till } \epsilon\text{-solution}}$$

| $k$ | $P(\mathcal{D})$ | $C(\mathcal{D})$ | $I(\mathcal{D})$ |
|---|---|---|---|
| 0 | $O(n)$ | $O(n)$ | $\frac{\mathrm{Tr}(\mathbf{A})}{\lambda_1}\ln(1/\epsilon)$ |
| $0<k<n-1$ | computation of $\lambda_i$ for $i=1,2,\dots,k+1$<br>computation of $u_i$ for $i=1,2,\dots,k$ | $O(n)$ | $\frac{C_k}{\lambda_{k+1}}\ln(1/\epsilon)$ |
| $n-1$ | computation of $\lambda_i$ for $i=1,2,\dots,n$<br>computation of $u_i$ for $i=1,2,\dots,n-1$ | $O(n)$ | $n\ln(1/\epsilon)$ |

## 5. Numerical Experiments

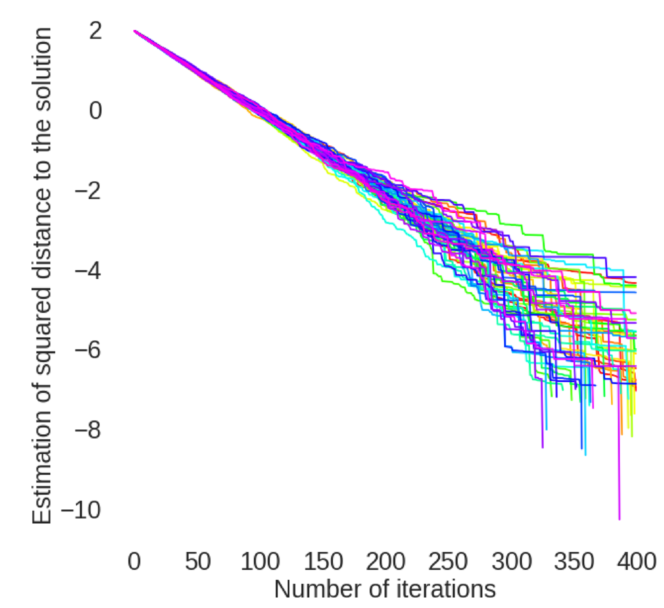


Figure: Eigenvalues were sampled from uniform distribution on $[10;11]$; $n=50$

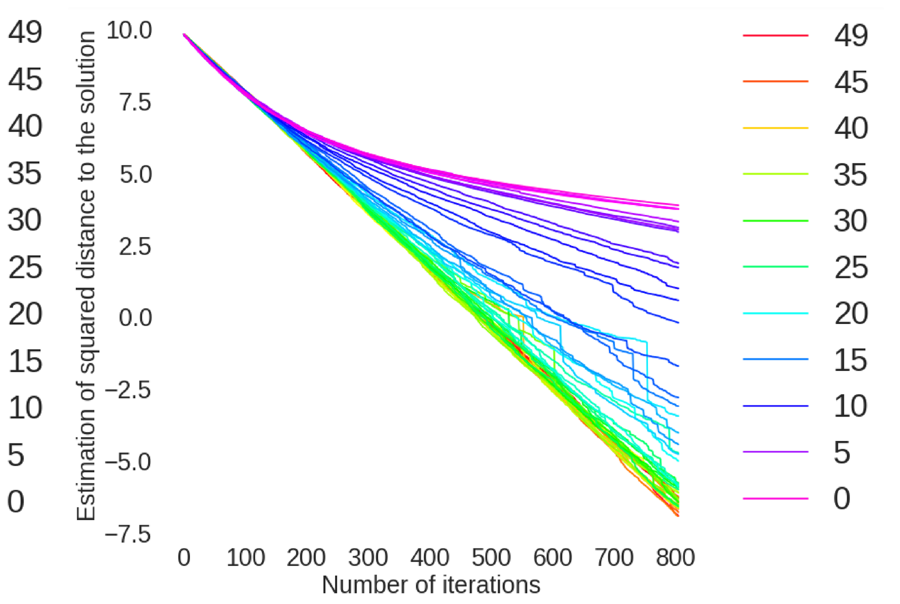


Figure: Eigenvalues were sampled from uniform distribution on $[0;100,000]$; $n=50$

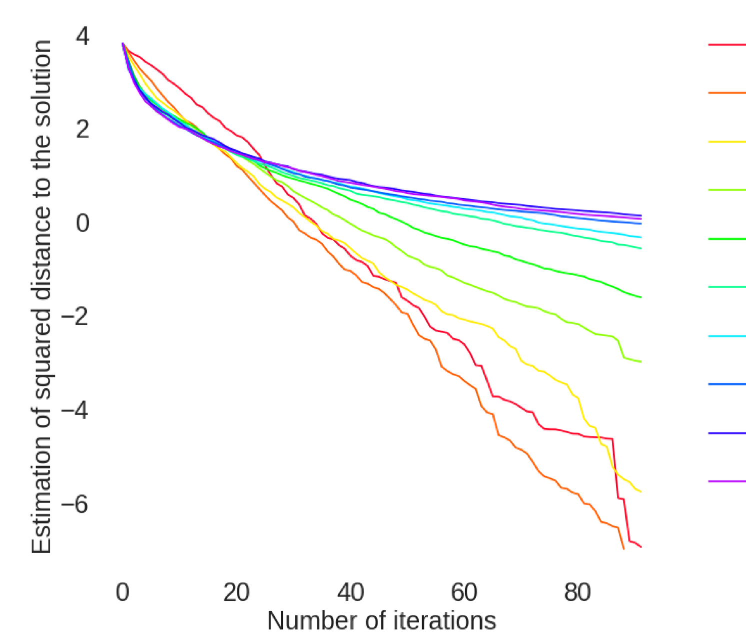


Figure: Eigenvalues decay exponentially; $n=10$

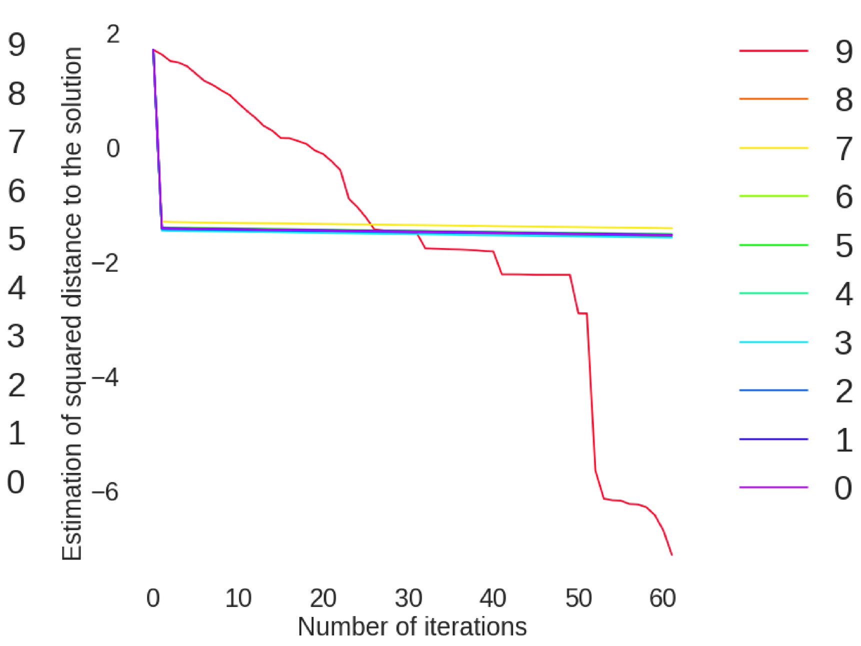


Figure: All eigenvalues equal to 1, except for the largest, which is equal to 1,000; $n=10$

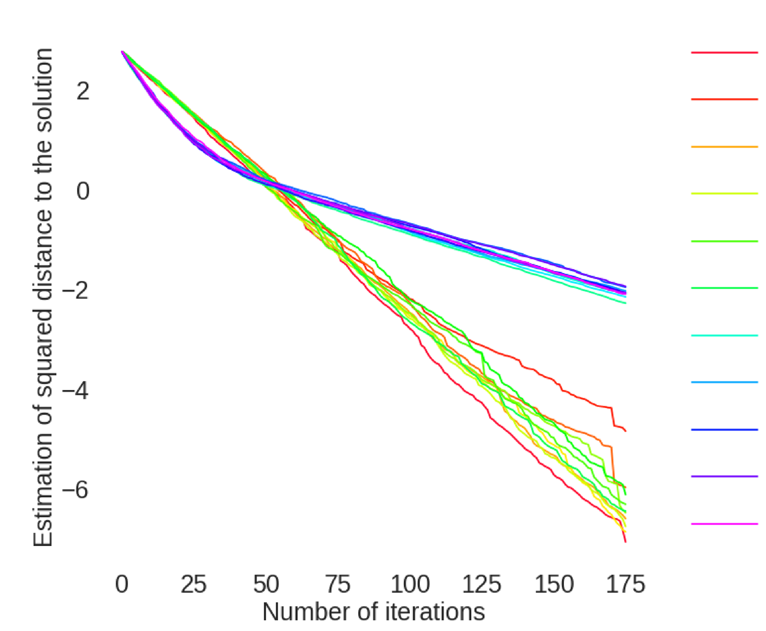


Figure: Half of eigenvalues were sampled from uniform distribution on $[10,11]$ and half from uniform distribution on $[100,101]$; $n=20$

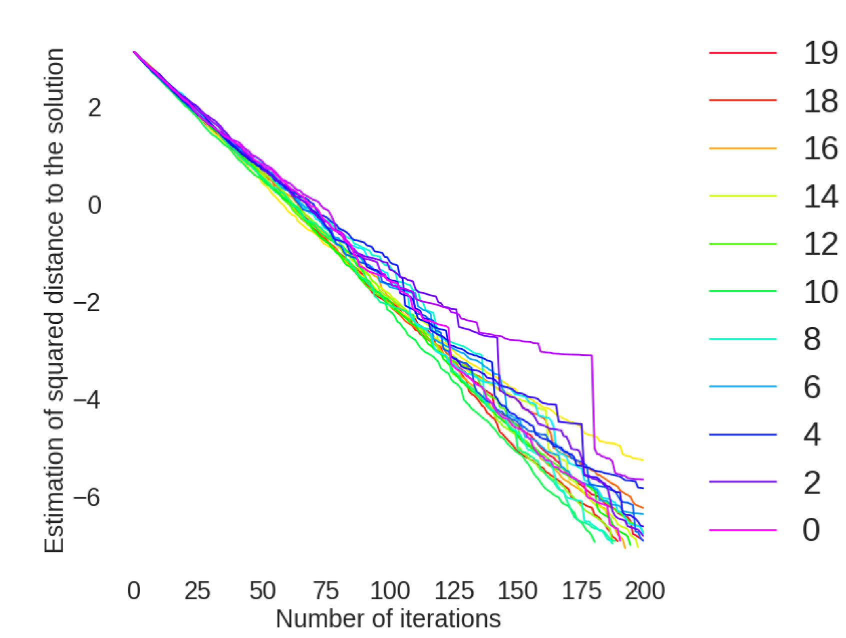


Figure: Half of eigenvalues were sampled from uniform distribution on $[50,51]$ and half from uniform distribution on $[100,101]$; $n=20$

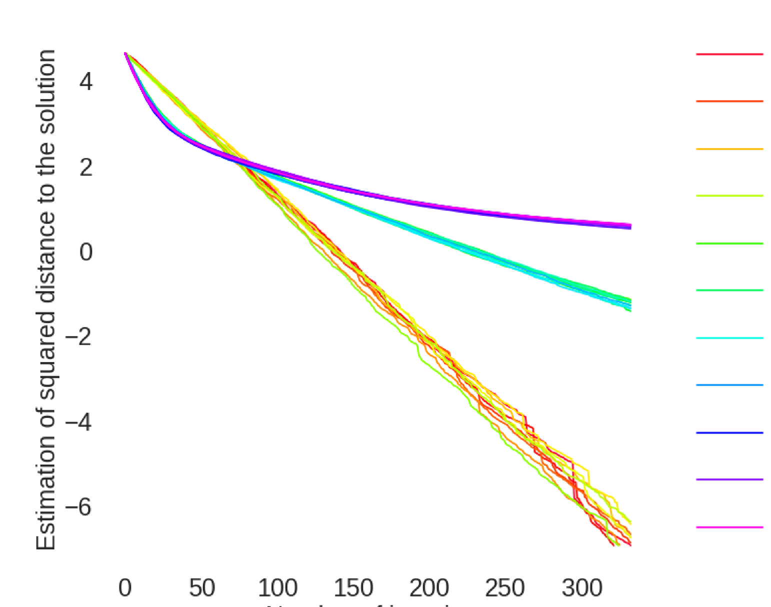


Figure: One third of eigenvalues were sampled from uniform distribution on $[10;11]$, one third from uniform distribution on $[100;101]$ and one third from uniform distribution on $[1,000;1,001]$; $n=30$

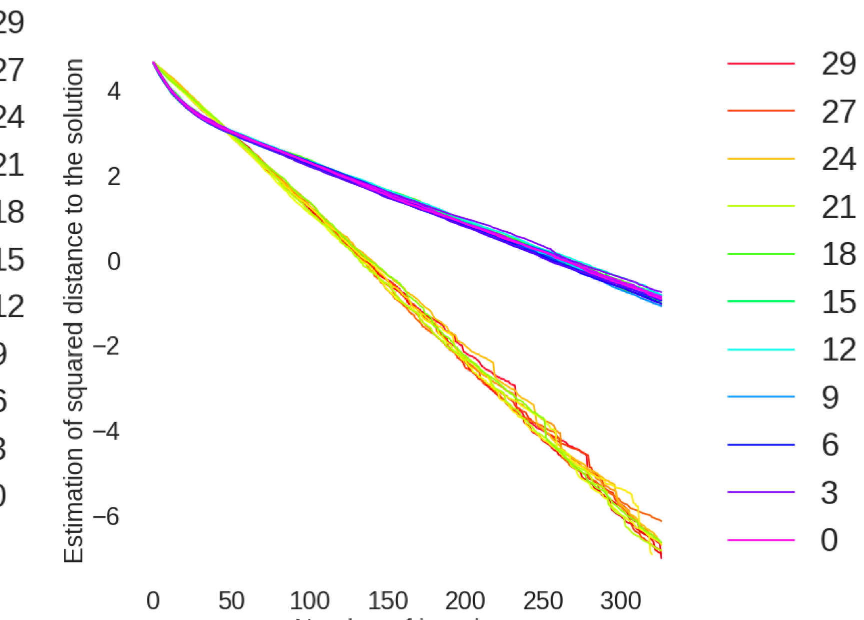


Figure: Two thirds of eigenvalues were sampled from uniform distribution on $[100;101]$ and one third from uniform distribution on $[1000,1001]$; $n=30$

## 6. Bibliography

[1] R. M. Gower and P. Richtárik. Randomized iterative methods for linear systems. *SIAM Journal on Matrix Analysis and Applications*, 36(4):1660–1690, 2015.

[2] D. Kovalev, E. Gorbunov, E. Gasanov, and P. Richtárik. Stochastic spectral and conjugate descent methods. *NIPS 2018*.

[3] P. Richtárik and M. Takáč. Stochastic reformulations of linear systems: Algorithms and convergence theory. *arXiv preprint arXiv:1706.01108*, 2017.